Part I. Compressible flow (cont'd)

3. Systems of conservation laws

Let's now turn our attention to systems of equations which define conservation laws in one spatial dimension. We would like to solve an initial-value problem for $U(x, t) \in \Re^N$ with $F(U) \in \Re^N$ such that:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{1}$$

with $U(x,0) = U_o$ given. We now know that, even for smooth U_o, the solution will develop discontinuities in finite time. So, we need a numerical method that will recognize and accommodate this limitation. For example, a weak solution will do:

$$\int_{x_L}^{x_R} [U(x, t^N) - U(x, t^0)]\, dx + \int_{t^0}^{t^N} [F(U(x_R, t)) - F(U(x_L, t))]\, dt = 0 \tag{2}$$

Note that U now represents a vector, with each component of U corresponding to a conserved quantity. For the case of gas dynamics, the conserved quantities are density ρ, momentum, ρu, and energy ρE:

$$U = (\rho, \rho u, \rho E)^T \quad F(U) = (\rho u, \rho u^2 + p, \rho u E + up)^T$$

The pressure p is assumed to be a function of density and internal energy e, where e is defined by:

$$e = E - \frac{u^2}{2} \tag{3}$$

so that pressure may be given by, for example:

$$p = \frac{\rho e}{\gamma - 1} \tag{4}$$

where γ is the ratio of specific heats at constant pressure to that at constant volume.

3.1 The linearized perturbation equations

Assume that U can be expressed as some base state, U_o, with a small perturbation around that state, U', i.e.,

$$U = U_o + U' \tag{5}$$

Let's assume that U_o is constant. We can substitute the expansion of equation (-1) into the PDE (-5):

$$\frac{\partial U'}{\partial t} + \frac{\partial F(U_o + U')}{\partial x} = 0$$

To write equation (-1) in linearized form, first use the definition of the gradient of F with respect to U to denote:

$$A(U_o) = \nabla_U F|_{U = U_o} = \left.\frac{\partial F_i}{\partial U_j}\right|_{U = U_o} \tag{6}$$

then we can expand F by

$$F(U_o + U') = F(U_o) + A(U_o)U' + \dots \tag{7}$$

Neglecting terms of higher order than the perturbation U', we can form the linearized perturbation equation:

$$\frac{\partial U'}{\partial t} + A(U_o)\frac{\partial U'}{\partial x} = 0 \tag{8}$$

After this point, we will drop the prime on U. Notice that $A(U_o)$ is a constant-valued matrix since the gradient of F is evaluated at the base state U_o. What conditions on A are necessary for us to have a prayer of solving this system of equations? One major boon would be for the system to be *hyperbolic*. In this case, A has n linearly independent right eigenvectors r_k:

$$Ar_k = \lambda_k r_k \tag{9}$$

where λ_k are real eigenvalues. This will allow us to diagonalize A and decouple the system of equations (-2). If you don't believe this is so, just watch this fancy footwork.

Define R to be a rectangular matrix made up of columns of n right eigenvectors:

$$R = (r_1, \dots, r_k, \dots, r_n) \tag{10}$$

Then by equation (-2)

$$AR = RD$$

or

$$R^{-1}AR = D \tag{11}$$

where D is the diagonal matrix comprised of the eigenvalues:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \tag{12}$$

First, premultiply equation (-6) by R^{-1} (remember that RR^{-1} is the identity matrix):

$$R^{-1}\frac{\partial U}{\partial t} + R^{-1}ARR^{-1}\frac{\partial U}{\partial x} = 0$$

Since R^{-1} is a constant, it can be taken inside the differentials. Also use equation (-2) above for D to obtain:

$$\frac{\partial (R^{-1}U)}{\partial t} + D\frac{\partial (R^{-1}U)}{\partial x} = 0$$

Now let's define $W = R^{-1}U = (w_1, \dots, w_k, \dots, w_n)$. We then obtain the system of equations:

$$\frac{\partial W}{\partial t} + D\frac{\partial W}{\partial x} = 0 \tag{13}$$

which can also be written as n uncoupled scalar advection equations:

$$\frac{\partial w_k}{\partial t} + \lambda_k\frac{\partial w_k}{\partial x} = 0 \tag{14}$$

which looks terribly friendly because we know so much about scalar advection equations. Note that the eigenvalues represent the wavespeed of the kth component of W. The initial conditions on W are

$$W(x, 0) = R^{-1}U(x, 0) \tag{15}$$

The idea is to work for as long as we can in U variables.

3.1.1 Godunov's method (first order)

The methodology for solving a scalar uncoupled system of equations like this one is very similar to the approach we used for a single scalar equation:

At each time step, we know how much conserved "stuff" is in each cell, i.e.

$$U_j^n \approx \frac{1}{\Delta x}\int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} U(x, n\Delta t)\, dx \tag{16}$$

We then:

- find the interpolation function $U_I(x)$ subject to the constraint that

$$U_j^n \Delta x = \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} U_I(x)\,dx; \tag{17}$$

- solve the PDE exactly along the cell edge;

$$\frac{\partial U_I^{n,n+1}}{\partial t} + A\frac{\partial U_I^{n,n+1}}{\partial x} = 0 \tag{18}$$

  where the double superscript indicates that this solution for $U_I(x)$ exists between time levels $n\Delta t$ and $(n+1)\Delta t$.

- use conservative finite differencing to interpolate the new solution onto the grid:

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}\left[F_{j-1/2} - F_{j+1/2}\right] \tag{19}$$

  where the flux F is given by:

$$F_{j+1/2} = AU_{j+1/2} \tag{20}$$

For first-order Godunov, we use a piecewise linear interpolation function; we can think of the function U as a sequence of constant states separated by jumps at the cell edges. We then solve the Riemann problem at the cell edges with the values in adjacent cells corresponding to the left and right states (depending on the sign of the wavespeed):

$$U(x,0) = \begin{cases} U_L \ldots (x<0) \\ U_R \ldots (x>0) \end{cases} \tag{21}$$

or

$$W(x,0) = \begin{cases} R^{-1}U_L \ldots (x<0) \\ R^{-1}U_R \ldots (x>0) \end{cases} ; \quad w_k(x,0) = \begin{cases} w_{k_L} \ldots (x<0) \\ w_{k_R} \ldots (x>0) \end{cases}$$

The exact solution to the hyperbolic equation for w_k given by equation (-8) above is

$$w_k(x,t) = w_k(x-\lambda_k t, 0) \tag{22}$$

Consider the case of an isolated wave with constant speed λ_k. In the Lagrangian reference frame, the wave propagates at a constant speed. What happens to U?

First, define a left-eigenvector matrix, L, analogous to R, which is made up of n rows of left eigenvectors which satisfy:

$$l_k A = \lambda_k l_k \tag{23}$$

so that L looks like:

$$L = R^{-1} = \begin{bmatrix} l_1 \\ \dots \\ l_n \end{bmatrix} \tag{24}$$

Find the left and right states for the Riemann solver:

$$U_L = \{ \begin{matrix} U_{j-1} \dots if \dots (\lambda_k > 0) \\ U_j \dots if \dots (\lambda_k < 0) \end{matrix} \tag{25}$$

and

$$U_R = \{ \begin{matrix} U_j \dots if \dots (\lambda_k > 0) \\ U_{j+1} \dots if \dots (\lambda_k < 0) \end{matrix} \tag{26}$$

Then solve the Riemann problem with:

$$U(x, 0) = \{ \begin{matrix} U_L \dots (x < 0) \\ U_R \dots (x > 0) \end{matrix} \tag{27}$$

so that

$$U(x, t) = U_L + \sum_{\lambda_k < \frac{x}{t}} \alpha_k r_k \tag{28}$$

where

$$\alpha_k = l_k (U_R - U_L) \tag{29}$$

We are now in a position to show that slapping U with l_k will yield the components of w that travel at speed λ_k.

$$l_k U(x, t) = l_k U_L + l_k \sum_{\lambda_{k'} < \frac{x}{t}} \alpha_{k'} r_{k'} \tag{30}$$

For $\lambda_k \geq x/t$:

$$\begin{aligned} l_k U(x, t) &= l_k U_L \\ &= w_{k_L} \end{aligned}$$

And for $\lambda_k < x/t$:

$$\begin{aligned} l_k U(x, t) &= l_k U_L + l_k (U_L - U_R) \\ &= l_k U_R \end{aligned}$$

$$= w_{k_R}$$

We can think of w_k as the amplitude of the traveling disturbance for the *k*th wave. The way to extract that amplitude from the "primitive" variables is to perform the vector multiplication $l_k U$. In this analysis, we would like to keep things in terms of U as far as possible, since there is no global analog like w_k for general nonlinear problems.

3.1.2 Perturbations to the Riemann problem

Let's now consider some small perturbations $\delta U_{L,R}$ to the Riemann problem about the base states $U_{L,R}$. Let

$$U_L^P = U_L + \delta U_L \tag{31}$$

$$U_R^P = U_R + \delta U_R \tag{32}$$

Pick some point *x/t*. If

$$l_k \delta U_L = 0 \ldots for \ldots \left(\lambda_k > \frac{x}{t}\right) \tag{33}$$

and

$$l_k \delta U_R = 0 \ldots for \ldots \left(\lambda_k < \frac{x}{t}\right) \tag{34}$$

then

$$U(x, t) = U^P(x, t) \tag{35}$$

Proof:

$$U^P(x, t) = U_L + \delta U_L + \sum_{\lambda_k < \frac{x}{t}} \alpha_k^P r_k \tag{36}$$

where

$$\alpha_k^P = l_k (U_R + \delta U_R - U_L - \delta U_L)$$

but $l_k \delta U_R = 0$ for $\lambda_k < x/t$ so that the above equation is

$$\alpha_k^P = l_k (U_R - U_L - \delta U_L)$$
$$= l_k (U_R - U_L) - l_k \delta U_L$$

Or, using the definition of α_k

$$\alpha_k^P = \alpha_k - l_k \delta U_L \tag{37}$$

Substituting this expression into equation (-2)

$$U^P(x,t) = U_L + \delta U_L + \sum_{\lambda_k < \frac{x}{t}} \alpha_k r_k - \sum_{\lambda_k < \frac{x}{t}} l_k \delta U_L r_k$$

and, using the fact that $l_k \delta U_L = 0$ for $\lambda_k > x/t$

$$U^P(x,t) = U_L + \delta U_L + \sum_{\lambda_k < \frac{x}{t}} \alpha_k r_k - \sum_k (l_k \delta U_L) r_k \tag{38}$$

But since

$$\begin{aligned}\sum_k (l_k \delta U_L) r_k &= \sum_k (L \delta U_L)_k r_k \\ &= RL\delta U_L\end{aligned}$$

By construction, $RL = I$, so that

$$\sum_k (l_k \delta U_L) r_k = \delta U_L \tag{39}$$

Putting equation (-1) into (-2)

$$U^P(x,t) = U_L + \delta U_L + \sum_{\lambda_k < \frac{x}{t}} \alpha_k r_k - \delta U_L$$

or finally

$$U^P(x,t) = U(x,t) \tag{40}$$

3.1.3 Return of first-order Godunov

If we define U^{RP} as the solution to the Riemann problem with left and right states as U_j^n and $U^n{}_{j+1}$, we can update the solution by

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x} [F_{j-1/2} - F_{j+1/2}] \tag{41}$$

where

$$F_{j+1/2} = A U_{j+1/2}^{RP} \tag{42}$$

Define for discrete values

$$w_{k_j}^n = l_k U_j^n \tag{43}$$

Then

$$w_{k_j}^{n+1} = \begin{cases} w_{k_j}^n + \dfrac{\Delta t}{\Delta x}\lambda_k (w_{k_{j-1}}^n - w_{k_j}^n) \ldots if \ldots (\lambda_k > 0) \\ w_{k_j}^n + \dfrac{\Delta t}{\Delta x}\lambda_k (w_{k_j}^n - w_{k_{j+1}}^n) \ldots if \ldots (\lambda_k < 0) \end{cases} \tag{44}$$

The second terms on the right represent the flux in and out of the cell. If you do not immediately believe this, hit equation (-4) with l_k and see what happens to the flux:

$$\begin{aligned} l_k F_{j+1/2} &= l_k A U_{j+1/2}^{RP} \\ &= \lambda_k l_k U_{j+1/2}^{RP} \\ &= \begin{cases} \lambda_k w_{k_j}^n \ldots if \ldots (\lambda_k > 0) \\ \lambda_k w_{k_{j+1}}^n \ldots if \ldots (\lambda_k < 0) \end{cases} \end{aligned}$$

3.1.4 Second-order Godunov

By now, the routine should be fairly clear. The steps which we must follow are:

- Interpolate for $U_I(x)$;
- Solve for $U_I(x)$ exactly;
  - Compute that left and right states at the cell edges;
  - Solve the Riemann problem;
- Update the solution onto the grid with conservative finite differencing.

The big difference between first- and second-order Godunov is the use of limiting. We interpolate in a piecewise linear fashion by:

$$U_I(x) = U_j^n + \frac{(x - j\Delta x)}{\Delta x}\Delta U_j \tag{45}$$

where ΔU_j is the finite-difference approximation to

$$\Delta U_j \approx \Delta x \left.\frac{\partial U}{\partial x}\right|_{j\Delta x} \tag{46}$$

when the solution is smooth and which includes limiting when it is not, similar to van Leer. In terms of w_k's:

$$\begin{aligned} w_{k_I}(x) &= l_k U_j^n + l_k \frac{(x - j\Delta x)}{\Delta x}\Delta U_j \\ &= w_{k_j}^n + \frac{(x - j\Delta x)}{\Delta x} l_k \Delta U_j \end{aligned}$$

$$= w_{k_j}^n + \frac{(x - j\Delta x)}{\Delta x} \Delta w_{k_j}$$

As for van Leer, the goal is to prevent wild oscillations in w_k. To calculate the limiter, we must define right, left and center $\alpha_{k,j}$. Let

$$\alpha_{k_j}^L = l_k (U_j^n - U_{j-1}^n) = w_{k_j}^n - w_{k_{j-1}}^n \quad (47)$$

$$\alpha_{k_j}^C = l_k (U_{j+1}^n - U_{j-1}^n) = w_{k_{j+1}}^n - w_{k_{j-1}}^n \quad (48)$$

$$\alpha_{k_j}^R = l_k (U_{j+1}^n - U_j^n) = w_{k_{j+1}}^n - w_{k_j}^n \quad (49)$$

Further define

$$\alpha_{k_j} = \left\{ \begin{array}{c} (sign\,(\alpha_{k_j}^C))\, min\,(2\left|\alpha_{k_j}^L\right|, \left|\alpha_{k_j}^C\right|, 2\left|\alpha_{k_j}^R\right|) \ldots if \ldots (\alpha_{k_j}^L \alpha_{k_j}^R > 0) \\ 0 \ldots otherwise \end{array} \right. \quad (50)$$

When the solution is smooth, we would expect $\left|\alpha_{k_j}^C\right|$ to be a minimum. To compute the left and right states, use an upstream-centered Taylor expansion:

$$U_{j+1/2_L}^{n+1/2} = U_j^n + \frac{\Delta t}{2}\frac{\partial U}{\partial t} + \frac{\Delta x}{2}\frac{\partial U}{\partial x} + \ldots$$

$$\approx U_j^n + \frac{1}{2}\left(I - A\frac{\Delta t}{\Delta x}\right)\Delta x \frac{\partial U}{\partial x}$$

so that

$$U_{j+1/2_L}^{n+1/2} = U_j^n + \frac{1}{2}\left(I - A\frac{\Delta t}{\Delta x}\right)\Delta U_j \quad (51)$$

$$U_{j+1/2_R}^{n+1/2} = U_{j+1}^n - \frac{1}{2}\left(I + A\frac{\Delta t}{\Delta x}\right)\Delta U_{j+1} \quad (52)$$

Using the above equations (-1), (-2) as input states, evaluated at *x/t = 0*, compute the solution to the Riemann problem, $U_{j+1/2}^{n+1/2}$. To compute the predictor/corrector for second-order Godunov on the w_k's, we would like to claim that

$$l_k U_{j+1/2}^{n+1/2} = w_{k_{j+1/2}}^{n+1/2} = \left\{ \begin{array}{l} w_{k_j}^n + \frac{1}{2}\left(1 - \lambda_k \frac{\Delta t}{\Delta x}\right)\Delta w_{k_j}^n \ldots if \ldots (\lambda_k > 0) \\ w_{k_{j+1}}^n + \frac{1}{2}\left(1 - \lambda_k \frac{\Delta t}{\Delta x}\right)\Delta w_{k_{j+1}}^n \ldots if \ldots (\lambda_k < 0) \end{array} \right.$$

However, if we premultiply equation (-2) for the left state above by l_k, we obtain:

$$l_k \left(U_{j+1/2_L}^{n+1/2} = U_j^n + \frac{1}{2}\left(I - A\frac{\Delta t}{\Delta x}\right)\Delta U_j\right) \quad (53)$$

$$w_{k_{j+1/2_L}}^{n+1/2} = w_{k_j}^n + \frac{1}{2}(1-\lambda_k \frac{\Delta t}{\Delta x}) l_k \Delta U_j \tag{54}$$

For $\lambda_k > 0$, this is precisely the Godunov predictor that we want.

3.1.4.1 Stability of the scheme

The stability of this scheme can be proved exactly as for scalars since we are dealing with a system of uncoupled scalar equations. We know that

$$l_k U_j^{n+1} = w_{k_j}^{n+1} \tag{55}$$

and that the w_k's satisfy

$$w_{k_j}^{n+1} = w_{k_j}^n + \frac{\Delta t}{\Delta x}(\lambda_k w_{k_{j-1/2}}^{n+1/2} - \lambda_k w_{k_{j+1/2}}^{n+1/2}) \tag{56}$$

3.1.4.2 Local truncation error

Assume that we know the exact solution evaluated at discrete points in space and time

$$U_{E_j}^n = U(j\Delta x, n\Delta t) \tag{57}$$

We further assume the solution is smooth. Let's analyze the case for $\lambda_k \neq 0$. The local truncation error, *LTE*, is

$$LTE = U_E^{n+1} - L(U_E^n) \tag{58}$$

where $L(U_E^n)$ is given by

$$L(U_E^n) = U_E^n + \frac{\Delta t}{\Delta x}(A\tilde{U}_{E_{j-1/2}} - A\tilde{U}_{E_{j+1/2}}) \tag{59}$$

In equation (-1), the term with the twiddle comes from evaluation of the Riemann problem with left and right states given by

$$\tilde{U}_{E_{j+1/2_L}} = U_{E_j}^n + \frac{1}{2}\left[I - A\frac{\Delta t}{\Delta x}\right](\Delta U_E^n)_j \tag{60}$$

$$\tilde{U}_{E_{j+1/2_R}} = U_{E_j}^n - \frac{1}{2}\left[I + A\frac{\Delta t}{\Delta x}\right](\Delta U_E^n)_j \tag{61}$$

We would like to prove, as for the Lax argument, that the following difference is the sum of a smooth function of order Δx^3 and higher-order terms:

$$\tilde{U}_{E_{j+1/2}} - U_{E_{j+1/2}}^{n+1/2} = C\left[((j+\frac{1}{2})\Delta x)\Delta x^2\right] + O(\Delta x^3) \tag{62}$$

It is easy to show that $\tilde{U}_{E_{j+1/2_L}} - U^{n+1/2}_{E_{j+1/2}}$ satisfied the Lax-Wendroff condition by keeping a few more terms of the Taylor expansion and using the PDE, as was done for the linear scalar equation previously. This will, however, be left as an exercise for the conscientious student (I've always wanted to say that. It feels just as pompous to write it as to read it). Note that the former term picks out the positive eigenvalues, while $\tilde{U}_{E_{j+1/2_R}}$ picks out the negative eigenvalues. We will state without belaboring the algebra that

$$\tilde{w}_{k_{j+1/2}} = \begin{cases} \tilde{w}_{k_{j+1/2_L}} \ldots for \ldots (\lambda_k > 0) \\ \tilde{w}_{k_{j+1/2_R}} \ldots for \ldots (\lambda_k < 0) \end{cases} \tag{63}$$

so that $\tilde{w}_{k_{j+1/2}} - w^{n+1/2}_{E_{j+1/2}}$ satisfies the Lax-Wendroff condition.

3.2 Effect of nonlinear change of variables

We would like to pose the question: how do the eigenvectors and eigenvalues transform under nonlinear changes of variable? To answer this question, we will first set up the mechanics for the general case. Then, as an illustration of why this is useful, we will examine the specific case of gasdynamics.

3.2.1 General case

Given the system of n equations in the vector *U:*

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{64}$$

where $A(U) = \nabla_U F$. Recall from our previous discussion that have n eigenvalues λ_k and right eigenvectors r_k. We would like to find a nonlinear change of variables *V(U)*. The derivatives can then be transformed by:

$$\frac{\partial U}{\partial t} = \nabla_V U \frac{\partial V}{\partial t} \tag{65}$$

$$\frac{\partial U}{\partial x} = \nabla_V U \frac{\partial V}{\partial x} \tag{66}$$

From the inverse function theorem

$$(\nabla_V U)^{-1} = \nabla_U V \tag{67}$$

Armed with this knowledge and festively adding new notation $A(U) = A^U$, we can rewrite equation (-4) as:

$$\nabla_V U \frac{\partial V}{\partial t} + A^U \nabla_V U \frac{\partial V}{\partial x} = 0$$

Premultiplying by $(\nabla_V U)^{-1}$

$$\frac{\partial V}{\partial t} + (\nabla_V U)^{-1} A^U \nabla_V U \frac{\partial V}{\partial x} = 0$$

If we define A^V by

$$A^V = (\nabla_V U)^{-1} A^U (\nabla_V U) \tag{68}$$

we seem to have painlessly obtained the equation:

$$\frac{\partial V}{\partial t} + A^V \frac{\partial V}{\partial x} = 0 \tag{69}$$

If we can easily find A^V, $r^V{}_k$, and $\lambda^V{}_k$, how can we figure out what $r^U{}_k$, and $\lambda^U{}_k$ are supposed to be? We must first think about what the r_k's tell us. Consider the propagation of an isolated wave. The eigenvalues represent the amplitude of that wave traveling in space. Changing our dependent variables does not change the physical wave speed; it only changes what we measure it to be. Therefore, it makes sense that

$$r_k^U = (\nabla_V U)\, r_k^V \tag{70}$$

or equivalently

$$(\nabla_V U)^{-1} r_k^U = r_k^V \tag{71}$$

Premultiply the definition of A^V in equation (-2) above by $r^V{}_k$:

$$A^V r_k^V = (\nabla_V U)^{-1} A^U (\nabla_V U)\, r_k^V$$
$$= (\nabla_V U)^{-1} A^U (\nabla_V U)\, (\nabla_V U)^{-1} r_k^U$$
$$= (\nabla_V U)^{-1} A^U r_k^U$$
$$= \lambda_k (\nabla_V U)^{-1} r_k^U$$
$$= \lambda_k r_k^V$$

The astonishing result is that the eigenvalues do not change on this transformation! If we can find some nonlinear transformation for which the eigenvalues are easy to find, we may save ourselves a lot of work.

One further note: If we are dealing with the perturbation equations in U which are linearized about the base state U_o, $A^U = A(U_o)$ and is constant. Likewise $A^V = A(V_o)$. If U_{TOT} is comprised of the base state and the perturbation quantity U:

$$U_{TOT}(x,t) = U_o + U(x,t) \quad (72)$$

then we define

$$V_{TOT}(x,t) = V_o + (\nabla_U V)\, U(x,t) + O(U^2) \quad (73)$$

or

$$V_{TOT}(x,t) \approx V_o + V(x,t) \quad (74)$$

3.2.2 Gasdynamics

A quick review follows.

The nonconservation forms of the governing equations of continuity, momentum and energy in one dimension are:

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \quad (75)$$

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad (76)$$

$$\frac{De}{Dt} + \frac{p}{\rho}\frac{\partial u}{\partial x} = 0 \quad (77)$$

where ρ is the density of the fluid medium, u is the velocity, e is the internal energy and the pressure p is given by a constitutive relation $p = p(\rho, e)$, e.g., $p = (\gamma - 1)\rho e$. Recall that the conserved quantities are density, momentum ρu, and energy ρE, and that the energy is related to the internal energy by

$$E = e + \frac{u^2}{2} \quad (78)$$

The material derivative is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \quad (79)$$

For this case, U is

$$U = \begin{bmatrix} \rho \\ u \\ e \end{bmatrix} \quad (80)$$

Compare to the form used previously

$$\begin{bmatrix} \rho \\ u \\ e \end{bmatrix}_t + A \begin{bmatrix} \rho \\ u \\ e \end{bmatrix}_x = 0 \tag{81}$$

where A results from linearization about the base state of ρ_o, u_o, e_o and the subscripts denote differentiation with respect to that variable. You can verify for yourself that

$$A(U) = \begin{bmatrix} u & \rho & 0 \\ \frac{1}{\rho} p_\rho & u & \frac{p_e}{\rho} \\ 0 & \frac{p}{\rho} & u \end{bmatrix} \tag{82}$$

We would now like to get from these variables to the "primitive" variables by a nonlinear transformation. We defined the U variables as:

$$U = (\rho, u, e)^T \tag{83}$$

We now define the transformed variables to be:

$$V(U) = (\rho, u, p)^T \tag{84}$$

We can relate the two by

$$V(U) = V(U_o) + \nabla_U V U' \tag{85}$$

How can we accomplish this change? The governing equations for density and velocity (-xx) and (-xxx) above do not change, but we need to do something different for the pressure. Since $p = p(\rho, e)$:

$$\frac{Dp}{Dt} = \left.\frac{\partial p}{\partial e}\right|_\rho \frac{De}{Dt} + \left.\frac{\partial p}{\partial \rho}\right|_e \frac{D\rho}{Dt}$$

by the chain rule. Denoting differentiation by subscripts

$$\frac{Dp}{Dt} = -p_e \frac{\dot{p}}{\rho^2} \rho \frac{\partial u}{\partial x} - p_\rho \rho \frac{\partial u}{\partial x}$$

Since the speed of sound is defined by

$$c^2 = p_e \frac{\dot{p}}{\rho^2} + p_\rho \tag{86}$$

the material derivative of the pressure can be written as

$$\frac{Dp}{Dt} = -c^2 \frac{\partial u}{\partial x} \tag{87}$$

or, using continuity

$$\frac{Dp}{Dt} = -\rho c^2 \frac{\partial u}{\partial x} \tag{88}$$

so, in V variables, the matrix A^V can be written as:

$$A^V = u\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \rho & 0 \\ 0 & 0 & \frac{1}{\rho} \\ 0 & \rho c^2 & 0 \end{bmatrix} \tag{89}$$

The eigenvectors of A^V can be easily found by considering only the eigenvectors of the second matrix on the r.h.s., since the first term on the r.h.s simply represents a shift in the propagation speed. This is the beauty of this change of variables; the second matrix on the r.h.s. reduces to a simple 2x2 because of the multitude of zeroes. Comparing this matrix to the one for A^U in equation (-xxx) above, we surmise that we could spend many nights dancing and frolicking if we can get away with this one.

The conscientious student can verify that the eigenvalues of both A^U and A^V are $u+c$, u, and $u-c$. Physically, if we think of an isolated blip as initial data, these eigenvalues correspond to three waves: one moves backward relative to the blip's speed, one moves with the disturbance, and one moves ahead of the perturbation. The right eigenvectors for A^V are given by:

$$R = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{c}{\rho} & 0 & \frac{c}{\rho} \\ c^2 & 0 & c^2 \end{bmatrix} \tag{90}$$